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AFDELING TOEGEPASTE WISKUNDE TW 149/75 MAY

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RELAXATION OSCILLATIONS GOVERNED BY A VAN DER POL EQUATION WITH PERIODIC FORCING TERM

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS(MOS) subject classification scheme (1970): 34C15, 34C25, 41A60

Relaxation oscillations governed by a van der Pol equation with periodic forcing term

by

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ABSTRACT

Asymptotic approximations are given for the period of a periodic solution of the inhomogeneous Van der Pol equation with a large parameter. The results are illustrated by many pictures and numerical integration techniques are used in order to verify the asymptotic methods.

KEY WORDS & PHRASES: *Inhomogeneous Van der Pol equation, relaxation oscillations, asymptotic approximation*

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1. INTRODUCTION

We consider periodic solutions of the inhomogeneous Van der Pol equation

(1) $$\frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = b \cos t$$

for large values of the parameter ν. For $b = 0$ a periodic solution with period

(2) $$T_0 = (3-2\log 2)\nu + O(\nu^{-1/3})$$

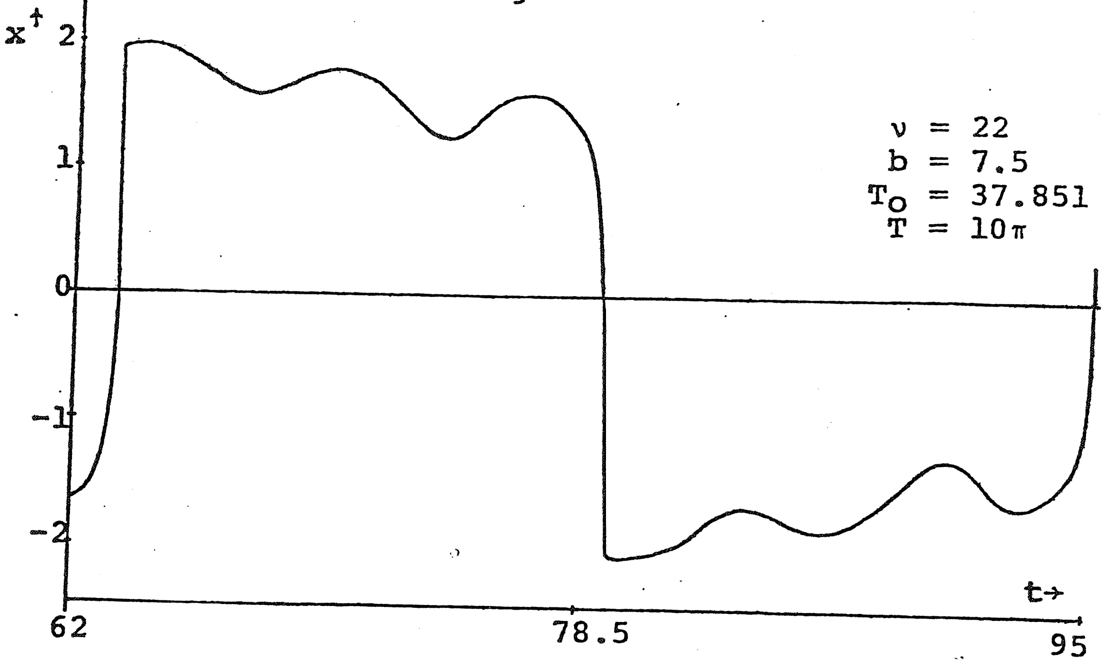
exists, see BAVINCK & GRASMAN [2] and COLE [3]. For (2) we write

(3) $$T_0 = 2\pi(2n+1) + 2\delta(\nu),$$

where $\delta(\nu) = O(1)$ and n is a large positive integer of $O(\nu)$. In this paper we will construct matched local asymptotic approximations for a periodic solution of (1) with period $T = 2\pi(2n+1)$, for $n = 2$ see figure 1. Moreover, necessary conditions will be given for the construction of such formal approximations. These conditions imposed on b and δ can be considered as formal conditions for synchronization. The method of matched asymptotic approximations we apply is related to a method of COLE [3] for solving the autonomous equation. For $b = 0$ our solution will differ at some points. However, these differences do not affect the solution of COLE quantitatively.

The local approximations mentioned above have different regions of validity, see figure 2. In one region (region A) the method of two time-scales is applied in order to obtain a solution that holds for large values of t. For other examples of this method the reader is referred to LAGERSTROM & CARSTEN [4]. For studying the local behavior of the solution in the other regions formal approximations are constructed as follows. The variables x and t are stretched according to a transformation of the type

Fig.1



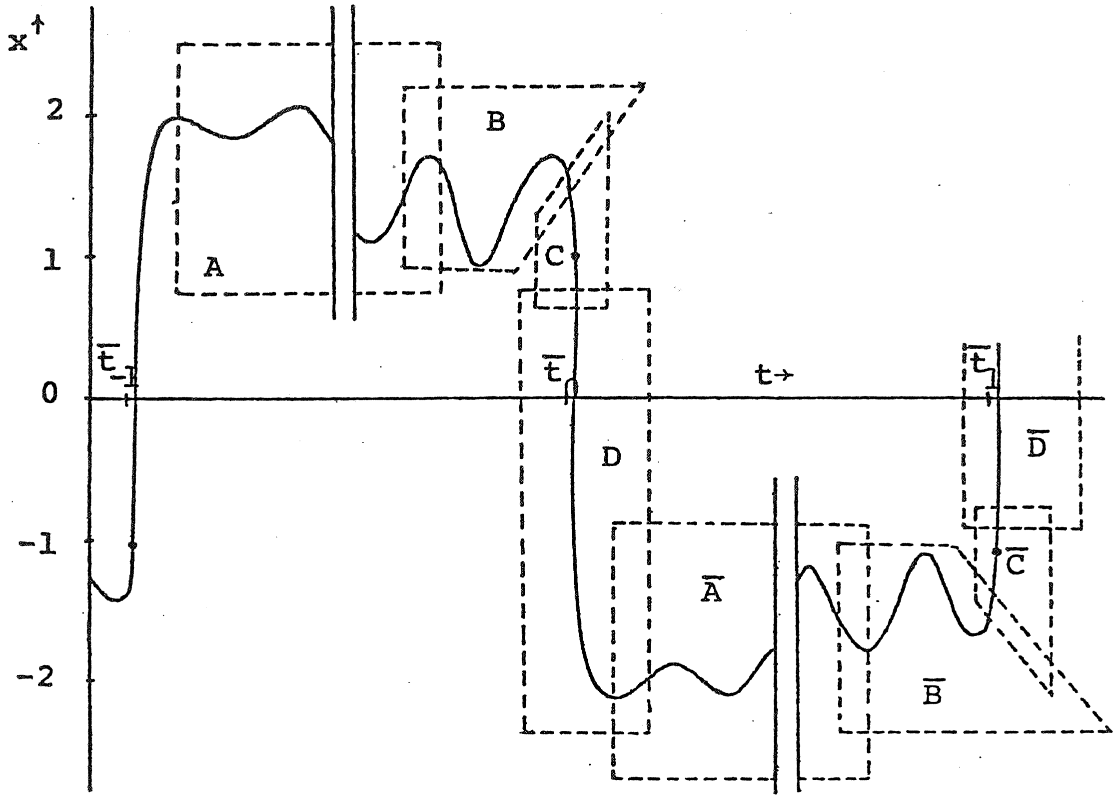


Fig. 2

(4) $$x = x_0 + \nu^{-\alpha} X(\xi,\nu), \qquad \alpha \geq 0,$$

(5) $$t = t_0 + \nu^{-\beta}\xi, \qquad \beta \geq 0.$$

Further (4) and (5) are substituted into equation (1), so that this equation transforms into

(6) $$\nu^{-\alpha+2\beta} \frac{d^2X}{d\xi^2} + \nu^{1-\alpha+\beta} \{(x_0 + \nu^{-\alpha}X)^2 - 1\} \frac{dX}{d\xi} + x_0 + \nu^{-\alpha}X = $$
$$= b \cos(t_0+\nu^{-\beta}\xi).$$

This equation in the local variables is multiplied with $\nu^{-\gamma}$, where

$$\gamma = \max\{-\alpha+2\beta,\ 1-\alpha+\beta,\ 0\} \quad \text{for } x_0 \neq 1,$$
$$\gamma = \max\{-\alpha+2\beta,\ 1-2\alpha+\beta,\ 0\} \quad \text{for } x_0 = 1,$$

then a so-called limiting equation is obtained by letting $\nu \to \infty$.
For revealing the local behavior of the solution of (1) a special choice of α and β has to be made, which will be worked out in the following sections. The solution of the limiting equation represents a formal local approximation of which the integration constants are determined by matching of two adjacent local approximations.

2. THE ASYMPTOTIC SOLUTION FOR REGION A

For region A we apply the method of two-time scales by introducing another independent variable $\tau = (t-t_0)/\nu$. It is supposed that the following expansion exists

(7) $$x = x_0(t,\tau) + x_1(t,\tau)\nu^{-1} + x_2(t,\tau)\nu^{-2} + \ldots \quad .$$

Substituting this expansion into equation (1) and equating terms of $O(\nu)$ we obtain the equation

$$(x_0^2-1)\frac{\partial x_0}{\partial t} = 0.$$

Hence the function $x_0(t,\tau)$ will only depend on τ Similarly from terms of $O(1)$ we get the equation

$$(x_0^2-1)\left(\frac{\partial x_1}{\partial t}+\frac{\partial x_0}{\partial \tau}\right)+x_0 = b\cos t.$$

This equation contains a slowly varying part

$$\hat{C}_1(\tau) \equiv (x_0^2-1)\frac{\partial x_0}{\partial \tau}+x_0$$

which would produce a secular term in the rapidly varying part of x_1, as

$$x_1(t,\tau) = \frac{b\sin t}{x_0^2-1} - \frac{\hat{C}_1(\tau)t}{x_0^2-1} + C_1(\tau).$$

Consequently $\hat{C}_1(\tau)$ must be taken identically zero, so that

$$\log x_0 - \tfrac{1}{2}(x_0^2-1) = \tau - D_0, \tag{8}$$

where D_0 denotes an integration constant. For $x_2(t,\tau)$ the following equation is found

$$\frac{\partial^2 x_1}{\partial t^2} + (x_0^2-1)\left(\frac{\partial x_2}{\partial t}+\frac{\partial x_1}{\partial \tau}\right) + 2x_0x_1\left(\frac{\partial x_1}{\partial t}+\frac{\partial x_0}{\partial \tau}\right) + x_1 = 0,$$

which has a slowly varying part

$$\hat{C}_2(\tau) = (x_0^2-1)\frac{\partial C_1}{\partial \tau} + 2x_0C_1\frac{\partial x_0}{\partial \tau} + C_1.$$

In order to remove secular terms in the rapidly varying part of x_2, we also take $\hat{C}_2(\tau)$ identically zero, so that

$$C_1(\tau) = C_1[x_0(\tau)] = \frac{D_1x_0}{x_0^2-1} .$$

The constants $D_0, D_1, \ldots$ denote shifts in τ. This is also achieved by writing

(9) $$t_0 = t_0^{(-1)}\nu + t_0^{(0)} + t_0^{(1)}\nu^{-1} + t_0^{(2)}\nu^{-2} + \dots$$

and by setting $D_0 = D_1 = \dots = 0$. Substitution of (9) into (8) shows that indeed these two solutions are equivalent. The functions $x_i(t,\tau)$, $i = 1,2,\dots$ are singular in $t = t_0$. When t approaches a neighborhood of t_0 of magnitude O(1) the first two terms behave as

$$x_0 \approx 1 + \{(t_0-t)/\nu\}^{\frac{1}{2}}, \quad x_1 \approx \tfrac{1}{2}\, b \sin t \,\{\nu/(t_0-t)\}^{\frac{1}{2}} .$$

3. THE ASYMPTOTIC SOLUTION FOR REGION B

In region B, where $t = t_0 + O(1)$, the solution will be of the type

(10) $$x = 1 + U(t,\nu)\nu^{-1/2}.$$

Substituting (10) into equation (1) and letting $\nu \to \infty$ we obtain the limiting equation

$$2U_0 \frac{dU_0}{dt} + 1 = b \cos t.$$

Integration yields

(11) $$U_0(t) = \sqrt{b \sin t + (t_0-t) + E_0} .$$

For $t_0-t \gg 1$ this solution is expanded as

$$U_0(t) = \sqrt{t_0-t} + \frac{b \sin t + E_0}{2\sqrt{t_0-t}} + \dots ,$$

so that it matches the solution of region A for $E_0 = 0$.

4. THE ASYMPTOTIC SOLUTION FOR REGION C

Let $t = \bar{t}_0 = \bar{t}_0^{(-1)}\nu + \bar{t}_0^{(0)} + \dots$ be the smallest root satisfying the equation

(12) $b \sin \bar{t}_0 + (t-\bar{t}_0) = 0.$

In a neighborhood of this point, where $x \approx 1$, equation (1) exhibits a turning-point behavior. We introduce the local coordinate ξ and the new dependent variable $V(\xi)$ by

$$t = \bar{t}_0 + \xi\nu^{-1/3} \quad \text{and} \quad x = 1 + V(\xi)\nu^{-2/3},$$

so that the corresponding limiting equation also will contain the term with the second derivative. This equation is found from the limit process described in section 1, it reads

$$\frac{d^2V_0}{d\xi^2} + \frac{dV_0^2}{d\xi} + d = 0, \qquad d = 1 - b\cos\bar{t}_0,$$

and it has a general solution of the form

$$V_0(\xi) = \Psi'(\xi)/\Psi(\xi),$$

$$\Psi(\xi) = \lambda_1 \mathrm{Ai}(-d^{1/3}\xi + d^{-2/3}\mu) + \lambda_2 \mathrm{Bi}(-d^{1/3}\xi + d^{-2/3}\mu).$$

The functions Ai(z) and Bi(z) denote so-called Airy functions, see ABRAMOWITZ & STEGUN [1]. It appears that $V_0(\xi)$ matches the solution of region B only if $\lambda_2 = \mu = 0$ and $d > 0$, as $U_0(\bar{t}_0+\xi\nu^{-1/3}) \approx (-d\xi)^{\frac{1}{2}}\nu^{-1/6}$ for $\xi \ll -1$.

5. ASYMPTOTIC SOLUTION FOR REGION D

The Airy function Ai(z) has its largest zero for $z = -\alpha$ with $\alpha = 2.338107$, so in $t = \bar{t}_0 + \xi_0\nu^{-1/3}$ with $\xi_0 = \alpha d^{-1/3}$ the asymptotic solution for region C will be singular; $V_0(\xi) \approx (\xi-\xi_0)^{-1}$ near that point. We introduce a new local variable η according to the transformation

(13) $$t = \bar{t}_0 + \xi_0 \nu^{-1/3} + \eta\nu^{-1},$$

so that the first two terms of (1) become of a same order of magnitude. Substitution in (1) and multiplication of this equation with ν^{-2} yield the following limiting equation as $\nu \to \infty$

$$\frac{d^2W_0}{d\eta^2} + (W_0^2-1)\frac{dW_0}{d\eta} = 0.$$

For matching the solution of region C we have the condition $W_0(\eta) = 1 + \eta^{-1}$ as $\eta \to -\infty$. This condition is satisfied by selecting solutions of the type

$$\frac{1}{1 - W_0} + \frac{1}{3}\log\frac{W_0 + 2}{1 - W_0} = -\eta + H_0.$$

The constant H_0 is found from matching W_0 with higher order terms of the asymptotic solution in region C. It turns out that H_0 should depend on log ν. However, we will not specify H_0 here as we do not need this result for describing synchronization.

For $\eta >> 1$ we have the estimate

$$W_0(\eta) = -2 + O(e^{-3\eta}).$$

6. MATCHING OF THE SOLUTIONS FOR THE REGIONS D AND $\bar{A}$

In region $\bar{A}$ the solution is written as

$$x = \bar{x}_0(\bar{\tau}) + \bar{x}_1(t,\bar{\tau})\nu^{-1} + x_2(t,\bar{\tau})\nu^{-2} + \ldots , \quad \bar{\tau} = (t-t_1)/\nu,$$

where $\bar{x}_0(\tau)$ satisfies the equation

(14) $$\log(-\bar{x}_0) - \frac{1}{2}(x_0^2-1) = \bar{\tau}.$$

while for $\bar{x}_i(t,\bar{\tau})$, $i = 1,2,\ldots$, a similar recurrent system of equations holds as for terms $x_i(t,\tau)$ of region A. Let

$$t_1 = t_1^{(-1)}\nu + t_1^{(0)} + O(\nu^{-1/3}),$$

then substitution of (13) and (15) into (14) yields the relation

$$t_1^{(-1)} - \bar{t}_0^{(-1)} = \frac{3}{2} - \log 2.$$

Moreover, for $\bar{x}_0$ and $\bar{x}_1$ we find

$$\bar{x}_0 = -2 + \frac{2}{3}(\bar{t}_0^{(0)} - t_1^{(0)})\ \nu^{-1} + O(\nu^{-4/3}),$$

$$\bar{x}_1 = \frac{1}{3}\, b \sin \bar{t}_0 + O(\nu^{-1/3}).$$

The matching condition is satisfied if

(17) $$t_1^{(0)} = \bar{t}_0^{(0)} + \frac{1}{2}\, b \sin \bar{t}_0,$$

since $x = W_0(\eta) + O(\nu^{-4/3})$ in region D, see COLE [3] (the influence of the inhomogeneous term of (1) is $O(\nu^{-2})$ in this region).

7. FORMAL CONDITIONS FOR SYNCHRONIZATION

Let for $t = \bar{t}_1$ the asymptotic solution for region $\bar{B}$ intersect with the line $x = -1$, then similar to (12) the relation

(18) $$b \sin \bar{t}_1 + (\bar{t}_1 - t_1) = 0$$

holds. Moreover, in case of symmetric solutions the following periodicity condition has to be satisfied

(19) $$\bar{t}_1 = \bar{t}_0 + (2n+1)\pi.$$

Using (2), (3), (9), (12) and (15-19) we obtain

(20) $$-\sin \bar{t}_0 = \sin \bar{t}_1 = \frac{2\delta}{3b} + O(\nu^{-1/3})$$

so that

$$t_1 - \bar{t}_1 = t_0 - \bar{t}_0 = \frac{2}{3}\delta + O(\nu^{-1/3}).$$

Equation (20) can be solved, if

(21) $$\frac{2}{3}\left|\frac{\delta}{b}\right| \le 1.$$

This condition agrees with the idea we have about synchronization.

The amplitude of the forcing term gives an upper bound for the detuning δ. For $b > 1$ the function $U_0(t)$ of (11) may vanish for some $\bar{t}_0$ with $\bar{t}_0 - \bar{t}_{-1} < \pi(2n+1)$, so that the oscillator cannot complete its period $T = 2\pi(2n+1)$. We therefore assume that for $b > 1$ the following condition is satisfied

(22) $$\arccos\left(\frac{1}{b}\right) + \pi + \arcsin\left(\frac{2}{3}\,\frac{\delta}{b}\right) + \frac{2}{3}\,\delta > \sqrt{b^2 - 1}.$$

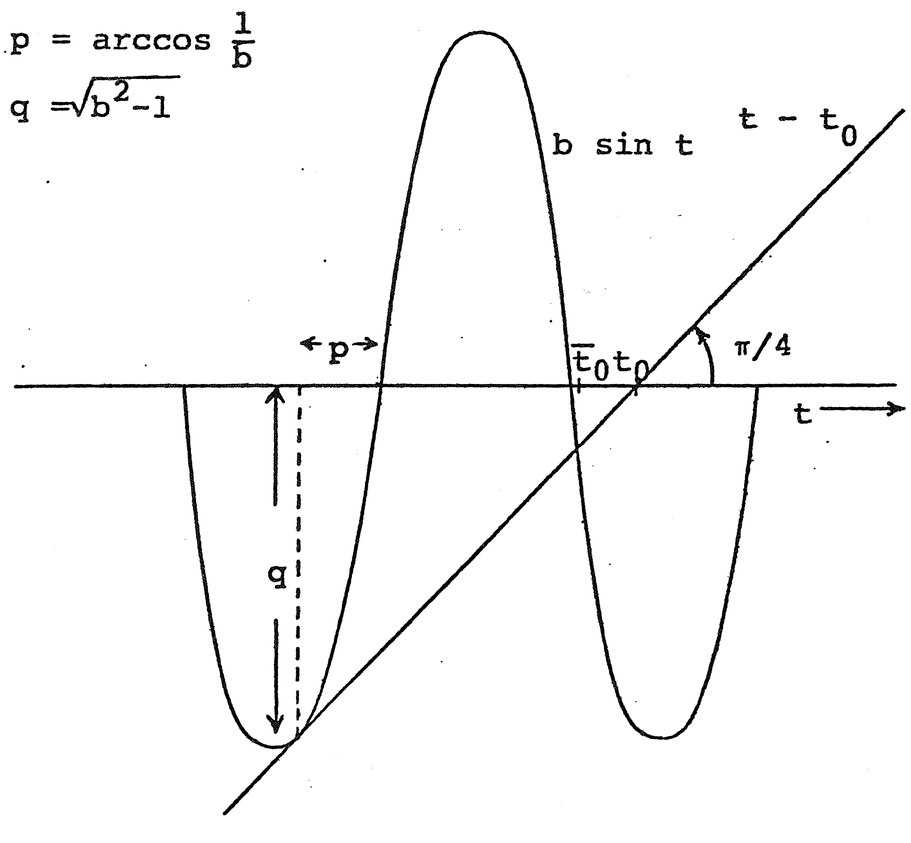


Fig. 3

In figure 3 it is shown that this condition can be derived from the limit case where a new zero of $U_0(t)$ smaller that $\bar{t}_0$ is about to arise. The solutions of (20) are

$$\bar{t}_0 = (2k+1)\pi + \arcsin\left(\frac{2}{3}\frac{\delta}{b}\right) + 0(\nu^{-1/3}), \tag{23}$$

$$\bar{t}_0 = 2k\pi - \arcsin\left(\frac{2}{3}\frac{\delta}{b}\right) + 0(\nu^{-1/3}), \quad k = 1,2,... \tag{24}$$

However, solution (24) does not satisfy the condition $d > 0$ of section 4. In figure 4 we sketch the domain where b and $\delta(\nu)$ satisfy the conditions (21) and (22). In this domain a synchronized periodic solution with period $T = 2\pi(2n+1)$, $n = 0(\nu)$ is possible.

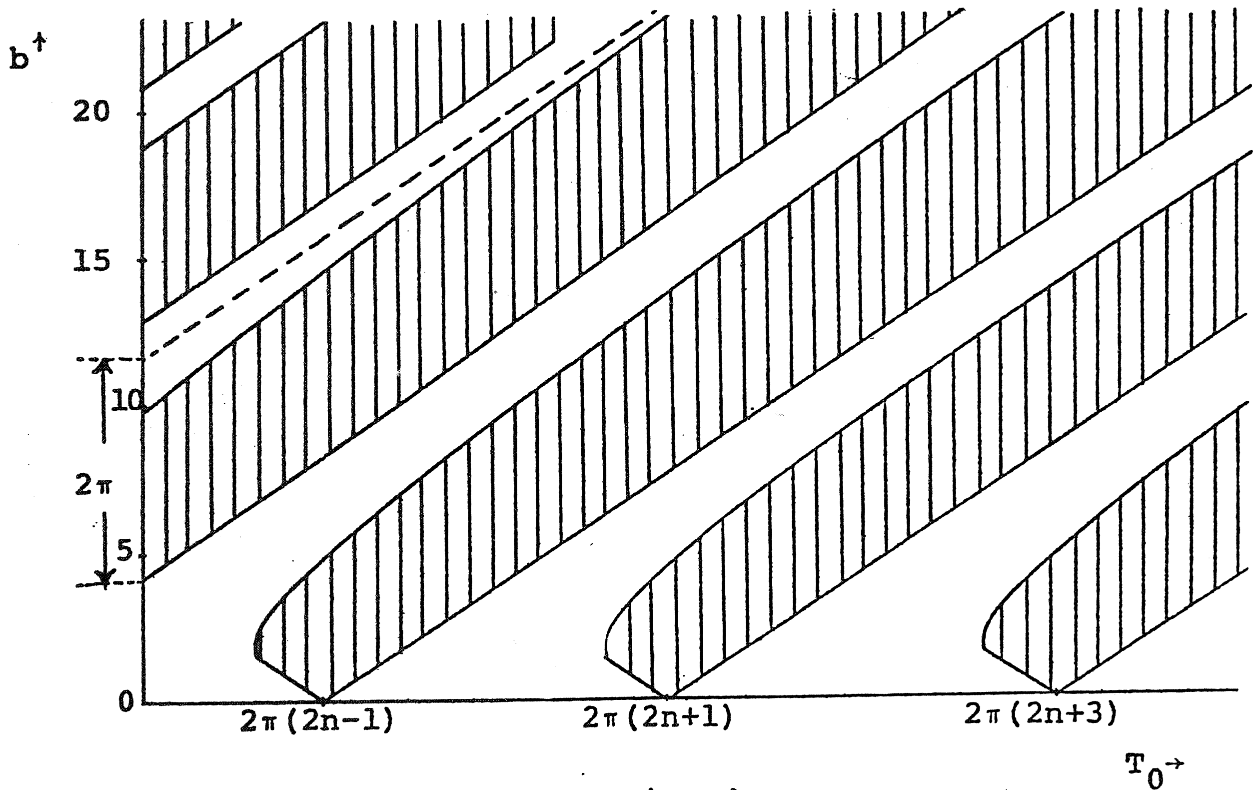


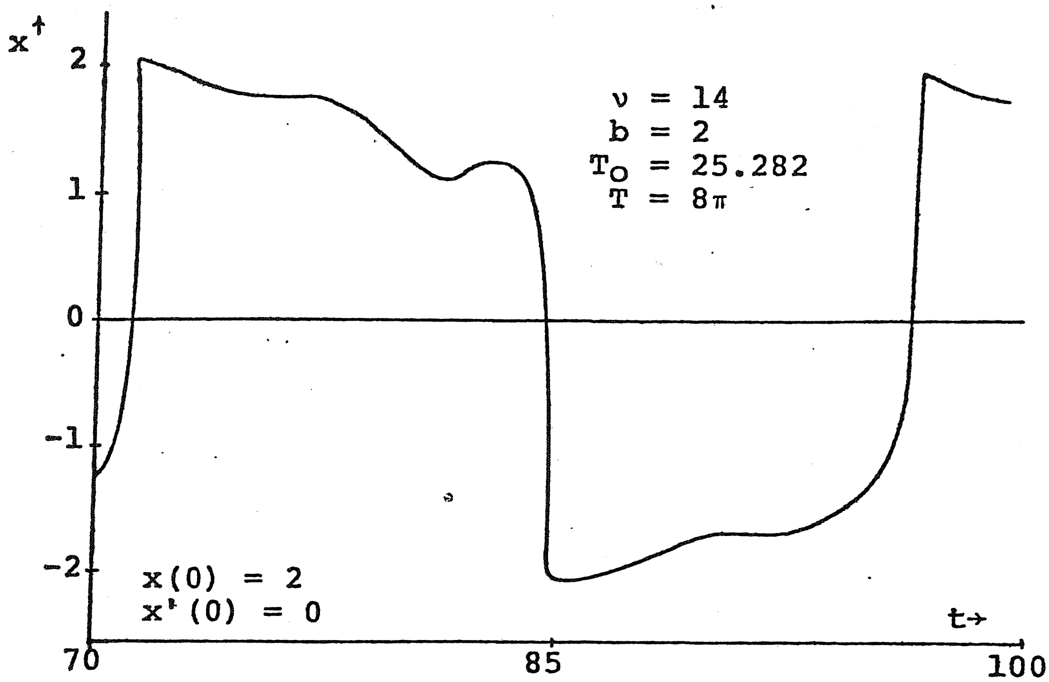
Fig. 4

8. SOME NUMERICAL RESULTS

A series of computer simulations has been made to compare the results of this formal asymptotic method with numerical solutions of (1) for different values of b and ν. Use has been made of a Runge-Kutta scheme (RK_{4na}, see ZONNEVELD [6]). For all simulations we took $x(0) = 2$ and $x'(0) = 0$. From other experiments we learned that in transition situations between the shaded domains of figure 4 the period may depend on the starting value. There also may arise periodic solutions with period $T = 4\pi n$, these are non-symmetric solutions (see fig.5) which are not considered in the asymptotic analysis.

We selected values around $\nu = 25$. For $\nu < 10$ the asymptotic solution does not hold anymore as we can see from figure 6. Moreover, for large values of b the computer results will differ from the asymptotic solution. Here we enter a region where the solution already is described qualitatively by topological-analytical methods, see LITTLEWOOD [5].

Fig. 5



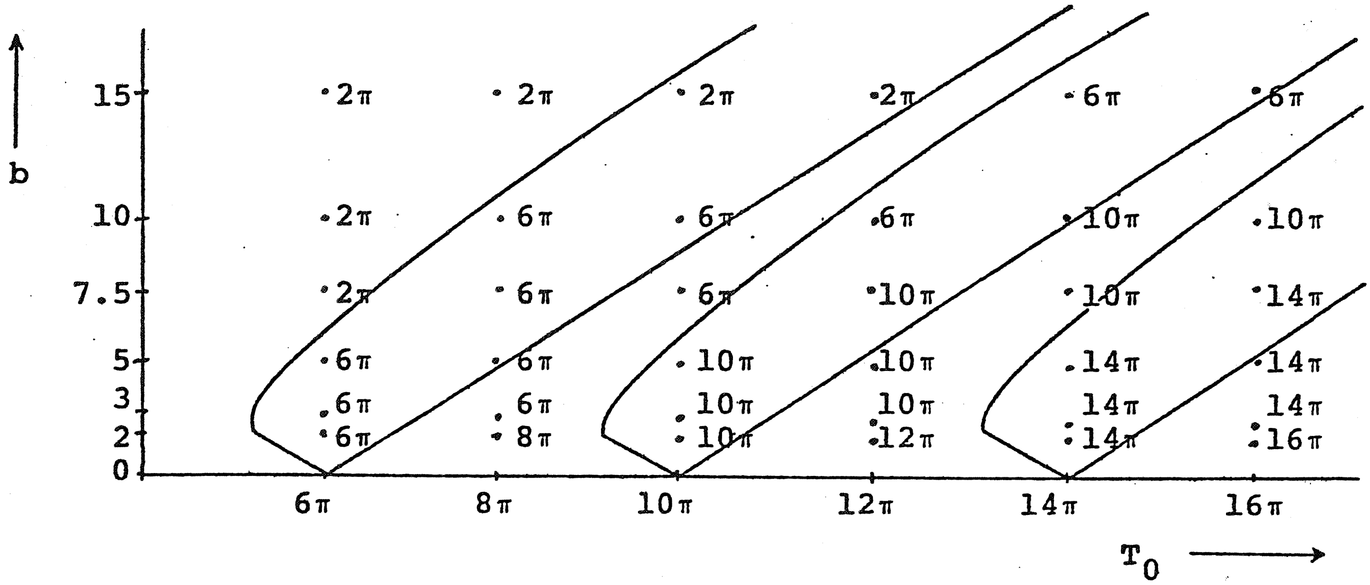


Fig.6

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